ON SELECTING THE CORRECT ROOT OF ANGLES-ONLY INITIAL ORBIT DETERMINATION EQUATIONS OF LAGRANGE, LAPLACE, AND GAUSS

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This paper is concerned with a classical yet still mystifying problem regarding multiple roots of the angles-only initial orbit determination (IOD) polynomial equations of Lagrange, Laplace, and Gauss of the form: $f(x) = x^8 + ax^6 + bx^3 + c = 0$ where $a, c < 0$. A possibility of multiple non-spurious roots of this 8th order polynomial equation with $b > 0$ has been extensively treated in the celestial mechanics literature. However, the literature on applied astrodynamics has not treated this multiple-root issue in detail, and not many specific numerical examples with multiple roots are available in the literature. Recently, Gim Der has claimed that the 200-year-old, angles-only IOD riddle associated with the discovery and tracking of asteroid Ceres has been finally solved by using a new angles-only IOD algorithm that doesn't utilize any *a priori* knowledge and/or additional observations of the object. In this paper, a very simple method of determining the correct root from two or three non-spurious roots is presented. The proposed method exploits a simple approximate polynomial equation of the form: $g(x) = x^8 + ax^6 = 0$. An approximate polynomial equation, either $g(x) = x^8 + c = 0$ or $g(x) = x^8 + ax^6 = x^6(x^2 + a) = 0$, can also be used for quickly estimating an initial guess of the correct root.

INTRODUCTION

It has been well known that the 8th order (or 8th degree) polynomial equations of Lagrange, Laplace, and Gauss can have at most three positive roots and thus the correct root must be determined from multiple positive roots. This paper is focused on such a classical, yet still mystifying, problem of determining the correct root of the 8th order polynomial equations of Lagrange, Laplace, and Gauss. Note that the term "correct root" herein doesn't really mean the "true" orbital range of a target object.

Recently, such a multiple-root problem of the angles-only initial orbit determination (IOD) polynomial equations was reexamined in [1–3]. It was claimed in [1–3] that the 200-year-old, angles-only IOD riddle associated with the discovery and tracking of asteroid Ceres has been finally solved by using a new angles-only IOD algorithm that does not utilize *a priori* knowledge and/or additional observations of the object. It was also claimed in [1–3] that a new range-solving angles-only algorithm can consistently determine the correct range and initial perturbed orbit of any unknown object in all orbit regimes without guessing and that this new algorithm allows modern optical sensors to be used for efficient and cost-effective catalog maintenance and catalog building. However,

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the details of this new algorithm for selecting the correct root from the multiple non-spurious roots are not available in open public domain.

A possibility of multiple non-spurious solutions of the classical 8th order polynomial equations of Lagrange, Laplace, and Gauss has been extensively discussed in the celestial mechanics literature [4–10]. However, the modern astrodynamics literature [11–15] have not treated this multiple-root issue in detail, while this multiple-root issue has been discussed in [16, 17] for angles-only space-based observations of satellites. Furthermore, not many specific numerical examples with multiple solutions are available in the literature, although numerical examples with a single positive root can be found in [13–15]. It has been well known that in case of multiple non-spurious solutions, *a priori* knowledge about the object and/or additional observations must be used for determining the correct range [4–10, 16, 17]. This paper is not concerned with comparing and/or improving the classical angles-only IOD methods of Laplace and Gauss. Such an IOD research problem of practical interest has been extensively discussed in the literature [18–23].

This paper is mainly focused on the fundamental yet practical problem of selecting the correct root from multiple non-spurious roots without a priori knowledge about the object and/or additional observations. The proposed simple solution presented in this paper will allow modern optical sensors to be used for efficient and cost-effective catalog maintenance and catalog building of orbital debris and/or near-Earth objects by contributing to real-time Space Situation/Domain Awareness.

LAGRANGE'S ANGLES-ONLY IOD PROBLEM FORMULATION

Three Angles-Only Observations

As illustrated in Figure 1 for the classical angles-only IOD problem [7, 11–14], the heliocentric position vector of a celestial object, $\mathbf{r}$, is described by

$$\mathbf{r} = \rho \mathbf{L} + \mathbf{R} \tag{1}$$

where $\rho\mathbf{L}$ is the geocentric position vector of a celestial object, $\mathbf{L}$ is the line-of-sight (LOS) unit vector, ρ is the slant range, and $\mathbf{R}$ is the gecentric vector from the center of Sun to the center of Earth. It is assumed that all these vectors are expressed in the SCI (Sun-Centered Inertial) reference frame. The LOS unit vectors at the three observation times are given by

$$\mathbf{L}_i = \begin{bmatrix} \cos\delta_i \cos\alpha_i \\ \cos\delta_i \sin\alpha_i \\ \sin\delta_i \end{bmatrix}, \quad i = 1, 2, 3 \tag{2}$$

where (α_i, δ_i) are the right ascension and declination of the object on the celestial sphere at an observation time t_i.

Because the LOS vector $\mathbf{L}$ traces out an orbital path on the celestial sphere, a set of orthonormal vectors $\{\mathbf{L}, \mathbf{T}, \mathbf{N}\}$ is defined such that

$$\mathbf{N} = \mathbf{L} \times \mathbf{T} \tag{3}$$

$$\mathbf{T} = \frac{d\mathbf{L}}{ds} \tag{4}$$

where $\mathbf{T}$ is the unit vector tangential to the orbital path as shown in Figure 1 and the arc length s is the distance measured along the path, and we have

$$\dot{s} \equiv \frac{ds}{dt} = \sqrt{\dot{\alpha}^2 \cos^2\delta + \dot{\delta}^2} \tag{5}$$

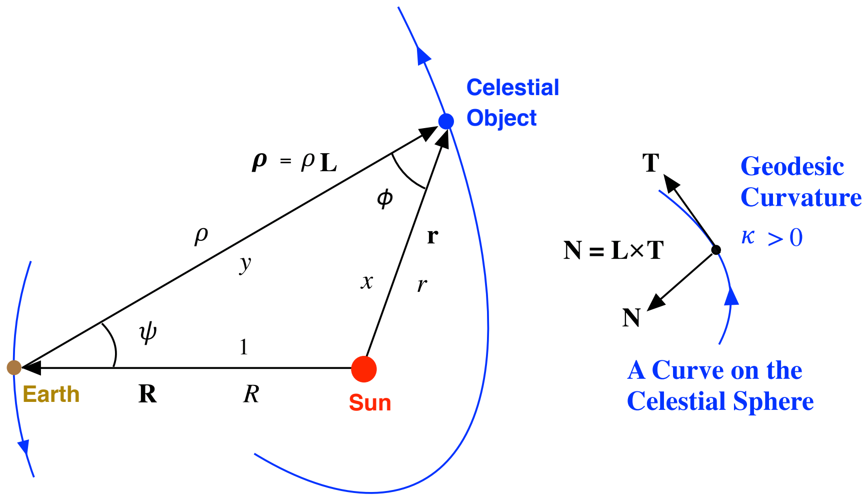

Figure 1. Lagrange's angles-only IOD problem.

The orbital path is then parametrized by using the geodesic curvature κ, as follows:

$$\frac{d\mathbf{T}}{ds} = -\mathbf{L} + \kappa\mathbf{N} \tag{6}$$

Using $\mathbf{r} - \mathbf{R} = \rho\mathbf{L}$, we obtain

$$\ddot{\mathbf{r}} - \ddot{\mathbf{R}} = \frac{d^2}{dt^2}(\rho\mathbf{L}) = (\ddot{\rho} - \rho\dot{s}^2)\mathbf{L} + (\rho\ddot{s} + 2\dot{\rho}\dot{s})\mathbf{T} + (\rho\dot{s}^2\kappa)\mathbf{N} \tag{7}$$

Taking a dot product of this equation by $\mathbf{N}$ results in

$$(\ddot{\mathbf{r}} - \ddot{\mathbf{R}}) \cdot \mathbf{N} = \rho\dot{s}^2\kappa \tag{8}$$

Because the orbital motions of the object and Earth are described by

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} \tag{9}$$

$$\ddot{\mathbf{R}} = -\mu\frac{\mathbf{R}}{R^3} \tag{10}$$

we obtain Lagrange's dynamical equation [7, 9] as

$$\mu\left(-\frac{1}{r^3} + \frac{1}{R^3}\right)\mathbf{R} \cdot \mathbf{N} = \rho\dot{s}^2\kappa \tag{11}$$

or

$$\begin{aligned} \rho &= \frac{\mu(\mathbf{R} \cdot \mathbf{N})}{\dot{s}^2\kappa}\left(\frac{1}{R^3} - \frac{1}{r^3}\right) \\ &= \frac{\mu(\mathbf{R} \cdot \mathbf{N})}{\dot{s}^2\kappa R^3}\left(1 - \frac{R^3}{r^3}\right) > 0 \end{aligned} \tag{12}$$

It is important to note that for a simple spherical surface of a celestial sphere, the geodesic curvature κ can be expressed as

$$\kappa = \frac{1}{\dot{s}^3} \det \begin{bmatrix} \mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{L}} \end{bmatrix} \tag{13}$$

and

$$\mathbf{R}\cdot\mathbf{N} = \frac{1}{\dot{s}}\det\left[\begin{array}{ccc}\mathbf{L} & \dot{\mathbf{L}} & \mathbf{R}\end{array}\right] \tag{14}$$

Consequently, Eq. (12) can be rewritten as

$$\rho = \frac{\mu}{R^3}\frac{\det\left[\begin{array}{ccc}\mathbf{L} & \dot{\mathbf{L}} & \mathbf{R}\end{array}\right]}{\det\left[\begin{array}{ccc}\mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{L}}\end{array}\right]}\left(1 - \frac{R^3}{r^3}\right) \tag{15}$$

which is in fact the same as Laplace's dynamical equation of the following form (as described in the Appendix):

$$\rho = \frac{\mu}{R^3}\frac{2D_2}{D}\left(1 - \frac{R^3}{r^3}\right) \tag{16}$$

where

$$\begin{aligned} D &= 2\det\left[\begin{array}{ccc}\mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{L}}\end{array}\right] \\ D_2 &= \det\left[\begin{array}{ccc}\mathbf{L} & \dot{\mathbf{L}} & \mathbf{R}\end{array}\right] \end{aligned}$$

From the Sun-Earth-Object triangle, we also obtain the geometric equation of the angles-only IOD problem as

$$r^2 = \rho^2 - 2R\rho\cos\psi + R^2 \tag{17}$$

Finally, Eqs. (12) and (17) can be solved for the two unknown r and ρ if $(\mathbf{R}\cdot\mathbf{N}) \neq 0$. By eliminating ρ, we obtain an 8th order polynomial equation in r (the heliocentric distance) of the object. One can easily notice that $r = R$ is a trivial solution of Eqs. (12) and (17), which corresponds to the orbit of the Earth. This trivial spurious solution ($r = R$ with $\rho = 0$) must be discarded.

The heliocentric IOD problem formulation of Lagrange as described in this section can also be directly applicable to the geocentric IOD problem with space-based or ground-based sensors, as illustrated in Figure 2. However, the ground-based geocentric IOD problem has no trivial spurious solution at $r = R$ because the ground-based observer is not in an orbit. However, the space-based geocentric IOD problem can have a trivial spurious solution at $r = R$ where R is the geocentric distance of a space-based sensor, which corresponds to the orbit of the space-based sensor.

Two classical methods of obtaining the 8th order polynomial equation in r are described in Appendix. They are the methods by Laplace and Gauss, which have been extensively treated in most textbooks on celestial mechanics and astrodynamics (e.g., [11, 13–15]).

THE 8TH ORDER POLYNOMIAL EQUATION

Descartes' Rule of Signs

As discussed in Appendix, the 8th order dimensionless polynomial equation of either Laplace or Gauss can be described by

$$f(x) = x^8 + ax^6 + bx^3 + c = 0 \tag{18}$$

where $x = r_2/R$ and r_2 is the magnitude of the position vector at an observation time t_2. Its positive roots provide preliminary orbits. The coefficients a, b, and c are computed using a given set of three angular observations, $\{(\alpha_i, \delta_i)$ at t_i, $i = 1, 2, 3\}$. However, the Gauss and Laplace equations will have different values of the coefficients (a, b, c) for a given set of three angular observations.

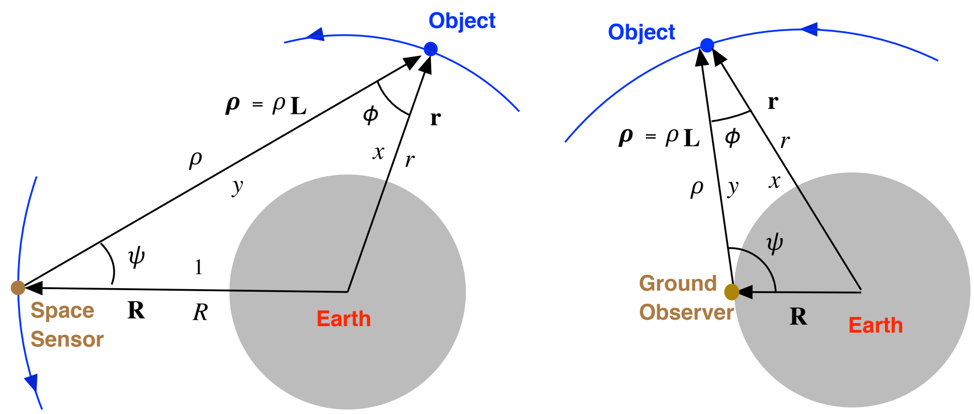

Figure 2. Geocentric geometry of the space-based or ground-based angles-only IOD problem.

As discussed in [1–10] using Descartes' rule of signs, the well-known properties of Eq. (18) can be summarized as follows:

(i) The coefficients a and c are always negative.

(ii) The coefficient b can be either negative or positive.

(iii) For $b < 0$, $f(x)$ has always one negative root and one positive root in addition to six complex roots.

(iv) For $b > 0$, $f(x)$ has one negative and *at most* three positive roots in addition to other complex roots; however, two positive roots are not possible because the 8th order polynomial equation must have an even number of real roots.

This paper is focused on selecting the correct root of case (iv) above with at most three positive roots.

Analytical Results by Charlier [4]

As illustrated in Figures 1, $r = |\mathbf{r}|$ denotes the heliocentric distance of a celestial object from the sun and ρ denotes its geocentric distance from the center of Earth. The two unknown, dimensionless variables (x, y) are introduced as

$$x = \frac{r}{R}, \quad y = \frac{\rho}{R} \tag{19}$$

where R is the orbital radius of Earth from the sun. Then, we obtain the geometrical constraint of the form

$$x^2 = 1 - 2y\cos\psi + y^2 \tag{20}$$

where ψ denotes the Sun-Earth-Object angle that is known from the observations as

$$\cos\psi = -\mathbf{L} \cdot \frac{\mathbf{R}}{R} \tag{21}$$

Equation (12) is rewritten as

$$y = \lambda\left(1 - \frac{1}{x^3}\right) \tag{22}$$

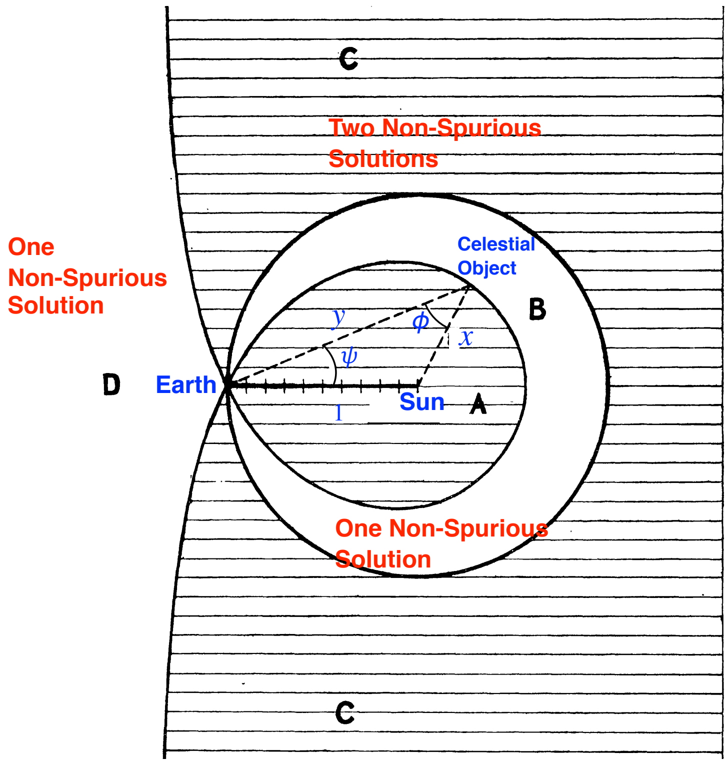

Figure 3. The dominion A or C for two non-spurious solutions and the dominion B or D for one correct solution [4].

where $x > 0$, $y > 0$, and λ is a constant determined from the observations. Combining Eqs. (20) and (22), we obtain an 8th order polynomial equation of the form

$$f(x) = x^8 - (1 + \lambda^2 - 2\lambda\cos\psi)x^6 + 2\lambda(\lambda - \cos\psi)x^3 - \lambda^2 = 0 \tag{23}$$

The coefficient of x^3 is always positive because

$$2\lambda(\lambda - \cos\psi) = \frac{\lambda^2}{x^3y^2}[(x+1)(x-1)^2(x^2+x+1) + (x^3+1)y^2] > 0 \tag{24}$$

One can easily verify that $x = 1$ is a trivial solution of Eq. (23), which corresponds to the orbit of the Earth. This trivial spurious solution must be discarded because $x = 1$ results in $y = 0$.

In [9], a positive root x of Eq. (23) is called a spurious solution if it produces a non-positive y for Eq. (22) and a positive root x of Eq. (23) is called a non-spurious solution if it produces a positive y for Eq. (22). Thus, the trivial solution $x = 1$ is a spurious solution to be discarded.

In [4], the condition for two non-spurious solutions has been stated, without a proof, as follows:

$$1 + 3\lambda\cos\psi > 0 \tag{25}$$

As discussed in [9], this condition can be obtained by considering $f(x)$ of the form

$$f(x) = (x-1)\tilde{f}(x) \tag{26}$$

where

$$\tilde{f}(x) = x^6(x+1) + (x^2+x+1)[\lambda^2 - (\lambda^2 - 2\lambda\cos\psi)x^3] \tag{27}$$

and

$$\begin{aligned}
&\tilde{f}(0) > 0 \\
&\tilde{f}(1) = 2 + 6\lambda\cos\psi \\
&\tilde{f}(\infty) = \infty
\end{aligned}$$

For a set of three positive roots ($x_1 < x_2 < x_3$), we can then conclude that if $\tilde{f}(1) < 0$, then $x_1 < 1 < x_3$ and if $\tilde{f}(1) > 0$, then we have either $x_1, x_2 < 1$ or $1 < x_2, x_3$.

The three different situations of a set of three positive roots (x_1, x_2, x_3) can then be summarized as

(i) If $x_1 = 1 < x_2, x_3$ and $\lambda > 0$, there are two non-spurious solutions x_2 and x_3.

(ii) If $x_1 < x_2 = 1 < x_3$, only one of (x_1, x_3) is the single correct solution; i.e., x_1 for $\lambda < 0$ or x_3 for $\lambda > 0$.

(iii) If $x_1, x_2 < x_3 = 1$ and $\lambda < 0$, there are two non-spurious solutions x_1 and x_2.

By substituting Eqs. (20) and (22) into Eq. (25), we obtain the condition for two non-spurious solutions as

$$1 > \frac{x^2 - 1 - y^2}{2} \frac{3}{1 - x^{-3}} \tag{28}$$

Two different dominions, illustrated in Figure 3, are separated by $x = 1$ (a sphere) and

$$y^2 = x^2 + \frac{2}{3x^3} - \frac{5}{3} \tag{29}$$

which is the equation of a surface of revolution about the Sun-Earth line.

As illustrated in Figure 3 [4], when an object is in the dominion A or C, there exists two non-spurious solutions (from three positive roots). When an object is in the dominion B or D, there is only one non-spurious solution.

Analytical Results by Danby [7]

In [7], a different approach to finding the condition for a single non-spurious solution or two non-spurious solutions was employed by using the following relationship of the Sun-Earth-Object triangle:

$$\frac{R}{\sin\phi} = \frac{r}{\sin\psi} = \frac{\rho}{\sin(\phi+\psi)} \tag{30}$$

where r, ρ, ϕ, and ψ are defined in Figure 1. Using the dynamical equation of the form

$$\rho = A\left(1 - \frac{R^3}{r^3}\right) > 0$$

we obtain

$$R\sin\psi\cos\phi + (R\cos\psi - A)\sin\phi = -A\frac{\sin^4\phi}{\sin^3\psi} \tag{31}$$

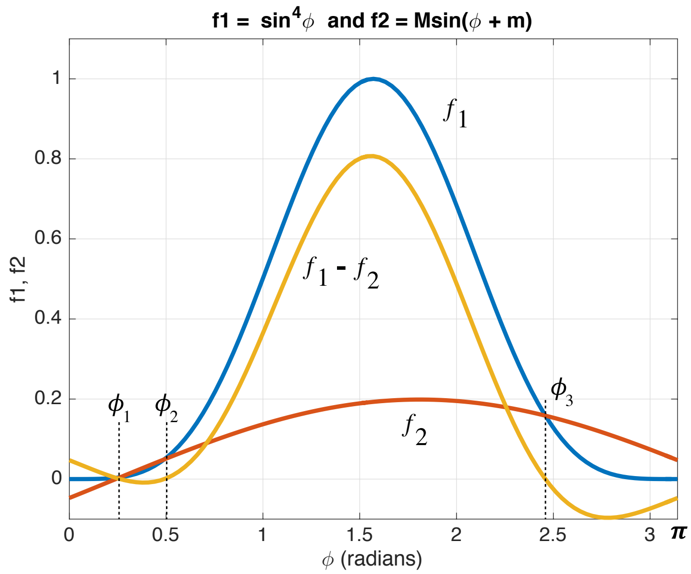

Figure 4. The three intersection points of two functions f_1 and f_2 [7, 16].

This equation can be simplified as

$$\sin^4 \phi = M \sin(\phi + m) \tag{32}$$

where M and m are defined such that

$$N \sin m = R \sin \psi \tag{33}$$

$$N \cos m = R \cos \psi - A \tag{34}$$

$$M = -\frac{N}{A} \sin^3 \psi \tag{35}$$

and the sign of N is chosen to make M positive. One can verify that $\phi = \pi - \psi$ is a trivial solution corresponding to the position of the observer, which must be discarded.

As illustrated in Figure 4 for a case with three solutions, the solutions of Eq. (32) can then be obtained from the intersections of the following two curves:

$$f_1(\phi) = \sin^4 \phi \tag{36}$$

$$f_2(\phi) = M \sin(\phi + m) \tag{37}$$

Once ϕ has been found, then r can be determined as

$$r = R \frac{\sin \psi}{\sin \phi} \tag{38}$$

Three different situations of three solutions (ϕ_1, ϕ_2, ϕ_3) can be summarized as

(i) If $\phi_1 = \pi - \psi < \phi_2, \phi_3$ and $A < 0$, there are two non-spurious solutions ϕ_2 and ϕ_3.

(ii) If $\phi_1 < \phi_2 = \pi - \psi < \phi_3$, either ϕ_1 for $A > 0$ or ϕ_3 for $A < 0$ is the single non-spurious solution.

(iii) If $\phi_1, \phi_2 < \phi_3 = \pi - \psi$ and $A > 0$, there are two non-spurious solutions ϕ_1 and ϕ_2.

Assuming three solutions as $\phi_1 < \phi_2 < \phi_3$, the condition for a single non-spurious solution was found in [7] as

$$\frac{\partial f_1}{\partial \phi} - \frac{\partial f_2}{\partial \phi} = 4\sin^3\phi\cos\phi - M\cos(\phi + m) > 0 \quad \text{at} \;\; \phi = \phi_2 = \pi - \psi \tag{39}$$

which becomes

$$-4\sin^3\psi\cos\psi + M\cos(\psi - m) > 0 \tag{40}$$

This expression can be further simplified to

$$\frac{4MA\cos\psi}{N} + \frac{M}{N}\left\{R\cos\psi\,(\cos\psi - A) + R\sin^2\psi\right\} > 0 \tag{41}$$

or

$$\frac{MR}{N}(1 + 3(A/R)\cos\psi) > 0 \tag{42}$$

Because $N < 0$ when $A > 0$ (vice versa, $N > 0$ when $A < 0$), we obtain the condition of a single non-spurious solution as

$$1 + 3(A/R)\cos\psi < 0 \tag{43}$$

or the condition for two non-spurious solutions as

$$1 + 3(A/R)\cos\psi > 0 \tag{44}$$

which is Charlier's Eq. (25) in terms of $\lambda \equiv A/R$.

In [16], the multiple-root issue also inherent to a geocentric IOD problem using angles-only space based observations of satellites was investigated employing a general dynamical equation of the form

$$\rho = A + \frac{B}{r^3} > 0 \tag{45}$$

which results in an 8th order polynomial equation of the form

$$f(r) = r^8 - (A^2 + 2AC + R^2)r^6 - 2B(A + C)r^3 - B^2 = 0 \tag{46}$$

where $C = \mathbf{L} \cdot \mathbf{R} = -R\cos\psi$.

For the general dynamical relationship described by Eq. (45), we also have

$$\sin^4\phi = M\sin(\phi + m) \tag{47}$$

where M and m are defined such that

$$N\sin m = R\sin\psi \tag{48}$$

$$N\cos m = R\cos\psi - A \tag{49}$$

$$M = \frac{NR^3}{B}\sin^3\psi \tag{50}$$

Thus, we notice that Eq. (47) is also applicable to the general dynamical relationship described by Eq. (45).

In [16], Eq. (47) is rewritten as

$$\sin^4\phi - M\sin\phi\cos m = M\cos\phi\sin m \tag{51}$$

Squaring this equation results in

$$\sin^8\phi - 2M\cos m\sin^5\phi + M^2\sin^2\phi - M^2\sin^2 m = 0 \tag{52}$$

From this 8th order polynomial equation in $\sin\phi$, we can also summarize the multiple-root problem, as follows:

(i) When $\cos m > 0$, there are one negative solution and at most three positive solutions.

(ii) When $\cos m < 0$, there are one positive solution and at most three negative solutions.

It is interesting to note that a figure identical to Charlier's dominion boundaries (Figure 3) can also be found in Danby [7]. However, in Danby [7], there is no mention about Charlier's 1910 paper. Vice versa, in Gronchi [8, 9], there is no mention about Danby's discussion of multiple solutions. According to [16], the approach described in this section first appeared in Watson [24], which was not mentioned in Danby [7].

Multiple-Root Issue

If a visible object lies above a ground-based observer's horizon, then we have

$$\frac{\pi}{2} < \psi < \pi \tag{53}$$

Consequently, in [1–3], the physically feasible correct ϕ and r ranges with good-quality observations are assumed to satisfy the following simple constraints:

$$0 < \phi < \pi - \psi < \frac{\pi}{2} \tag{54}$$

$$R_\oplus < r \tag{55}$$

where $R_\oplus = 6378.137$ km. It was also stated in [1–3] as: "For an orbiting space sensor of geocentric IOD problem (and a heliocentric IOD problem in celestial mechanics), at most two other roots may exist, but they are associated with $\phi > \pi/2$. By not allowing the line of sight to see through the Earth, one of the two possible roots can often be eliminated. Other intuitive relations may be needed."

It has been well known that once multiple non-spurious roots have been found, use of a priori knowledge, additional observations, and/or an assessment of each preliminary orbit must be used to determine the true correct root. However, it was claimed in [1–3] that the 200-year-old, angles-only IOD riddle has been finally solved by using a new angles-only IOD algorithm that does not utilize *a priori* knowledge and/or additional observations of the object. It was claimed in [1–3] that the new range-solving angles-only algorithm can consistently determine the correct range and initial perturbed orbit of any unknown object in all orbit regimes without guessing. Because the details of this new algorithm are not available in open public domain, we have attempted to find our own simple way of selecting the correct root, as presented in the next section.

Simple Approximation of the 8th Order Polynomial Equation

As described in Appendix, the 8th order IOD polynomial equation can be expressed as

$$f(x) = x^8 + ax^6 + bx^3 + c = 0 \tag{56}$$

where $x = r/R$, $y = \rho/R$, $a = -(A^2+2AC+R^2)/R^2$, $b = -2B(A+C)/R^5$, and $c = -B^2/R^8$. We also have the dimensionless dynamical equation of the angles-only IOD problem as

$$y = \tilde{A} + \frac{\tilde{B}}{x^3} \tag{57}$$

where $\tilde{A} = A/R$ and $\tilde{B} = B/R^4$.

Ignoring the gravitational terms of the Lagrange coefficients in Eqs. (90) and (91) of the Appendix, we obtain the zeroth-order (short-arc, straight-line motion) approximation of the Lagrange coefficients as

$$f_1 \approx 1; \quad g_1 \approx \tau_1 \tag{58}$$

$$f_3 \approx 1; \quad g_3 \approx \tau_3 \tag{59}$$

which results in an approximate 8th order polynomial equation of the form

$$f(x) \approx g(x) = x^8 + ax^6 = x^6(x^2 + a) = 0 \tag{60}$$

When the differences in measurement times (τ_1, τ_3) are very small compared to the orbital period of the object, an initial estimate of the correct root can be easily obtained as

$$x \approx \sqrt{|a|} \tag{61}$$

Conversely, if the gravitational terms in Eqs. (94) and (95) of the Appendix are more significant than the other zeroth-order terms (a_1, a_3), we have

$$\rho = A + \frac{B}{r^3} \approx \frac{B}{r^3} \tag{62}$$

and

$$f(x) \approx x^8 - x^6 - (2BC/R^5)x^3 - B^2/R^8 = 0 \tag{63}$$

This can be further approximated as

$$f(x) \approx g(x) = x^8 + c = 0 \tag{64}$$

if $|c| = B^2/R^8$ is very large compared to the other coefficients $|a|$ and $|b|$. This simple approximation can be used for long-arc observations of an object, especially in a circular orbit, to obtain an initial estimate of the correct root as

$$x \approx |c|^{1/8} \tag{65}$$

Furthermore, the positive root of $g(x) = x^8 + ax^6 = 0$ can be used to select the correct root from a set of multiple non-spurious roots of $f(x)$. The proposed method described herein, utilizing an approximate polynomial equation of the form: $g(x) = x^8 + ax^6 = 0$ but without requiring *a priori* knowledge and/or additional observations of the object, will be demonstrated in the next section.

NUMERICAL RESULTS

Our numerical evaluation of 10 sample examples of [2] indicate that most of them have a single positive root. Only two examples in [2] were found to have three positive roots for x. They are: Example 8 (Saturn) with three non-spurious roots and Example 10 (Ceres) with only one non-spurious root. For the examples in [2], employing the dimensionless orbital distance $x = r/R$, the actual orbital position r needs to be appropriately scaled using R = 149,597,870.691 km (for the heliocentric IOD) or R = 6378.137 km (for the geocentric ground-based IOD).

In this section, we consider various examples with $b > 0$. All roots of each example, including such meaningless complex roots as well as a negative root determined by Matlab, are provided below. The numerical values of the coefficients (a, b, c) for Examples 3, 8, and 10 provided in [2] are directly used for computing the roots. However, the numerical values of ($\tilde{A}$, $\tilde{B}$) are our own values computed from the three sets of observation data provided in [2].

Example 3 GEO [2] a = -0.6590; b = 343.975; c = -3506478.8 (Gauss coefficients)
($\tilde{A} = -0.6815$; $\tilde{B} = 1872.55$)
-6.613817101243090 + 0.000000000000000i
-4.637332673998804 + 4.642728797034024i
-4.637332673998804 - 4.642728797034024i
6.567720281040938 + 0.000000000000000i (correct root since $y = 5.9283 > 0$)
4.683255782596715 + 4.642555068251039i
4.683255782596715 - 4.642555068251039i
-0.022874698496830 + 6.565803624171794i
-0.022874698496830 - 6.565803624171794i

Example 8 Saturn [2] a = -94.2131; b = 176.2090; c = -83.2408 (Gauss coefficients)
($\tilde{A} = 9.7385$; $\tilde{B} = -9.1235$)
-9.716239069340666 + 0.000000000000000i
9.696386702677536 + 0.000000000000000i (correct root since $y = 9.7284 > 0$)
0.984335670856992 + 0.000000000000000i (non-spurious root since $y = 0.1724 > 0$)
0.980448242055782 + 0.000000000000000i (non-spurious root since $y = 0.0582 > 0$)
-0.517111466939649 + 0.817579495157434i
-0.517111466939649 - 0.817579495157434i
-0.455354306185167 + 0.878021461501731i
-0.455354306185167 - 0.878021461501731i

Example 10 Ceres [2] a = -8.3070; b = 7.4193; c = -1.6649 (Gauss coefficients)
($\tilde{A} = 1.9070$; $\tilde{B} = -1.2903$)
-2.933191182389097 + 0.000000000000000i
2.825106617751868 + 0.000000000000000i (correct root since $y = 1.8941 > 0$)
0.858226912758981 + 0.000000000000000i (spurious root since $y = -0.1341 < 0$)
0.713087068876388 + 0.000000000000000i (spurious root since $y = -1.6514 < 0$)
-0.327451190560088 + 0.713983683745073i
-0.327451190560088 - 0.713983683745073i
-0.404163517938981 + 0.607241293576432i
-0.404163517938981 - 0.607241293576432i

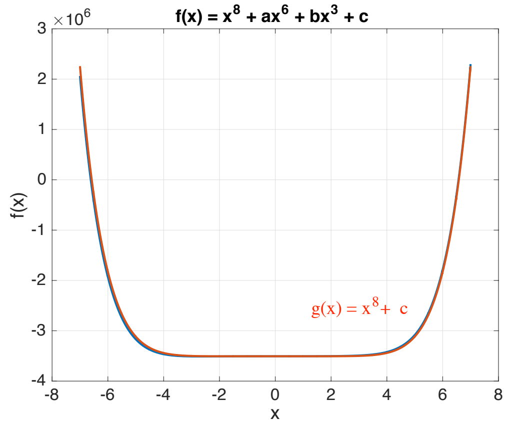

Figure 5. Example 3 GEO with Gauss coefficients [2].

Example 10 Ceres [2] a = -14.8667; b = 22.2634; c = -7.0259 (Laplace coefficients)
-3.904201576583076 + 0.000000000000000i
3.803197449981154 + 0.000000000000000i (correct root)
1.059854840360126 + 0.000000000000000i (spurious root since $y = 0$)
0.760368462240121 + 0.000000000000000i (spurious root since $y < 0$)
-0.467872452287063 + 0.877007575058880i
-0.467872452287063 - 0.877007575058880i
-0.391737135712101 + 0.663925421990250i
-0.391737135712101 - 0.663925421990250i

Danby [7] with a = -11.38616; b = 17.01671; c = -6.73854 (Laplace coefficients)
$\tilde{A} = 2.703$; $\tilde{B} = -2.596$
-3.436503819499419 + 0.000000000000000i
3.304506245794778 + 0.000000000000000i (correct root since $y = 2.631 > 0$)
0.986610358753644 + 0.000000000000000i (spurious root since $y = 0$)
0.891273733779891 + 0.000000000000000i (spurious root since $y = -0.9636 < 0$)
-0.508735144724709 + 0.721024434248609i
-0.508735144724709 - 0.721024434248609i
-0.364208114689742 + 0.856714837462722i
-0.364208114689742 - 0.856714837462722i

Plots of $f(x)$ in blue and $g(x) = x^8 + c$ in red for Example 3 (GEO) of [2] is shown in Figure 5. As can be seen in this figure, $g(x)$ is an excellent approximation of $f(x)$, which can be used for quickly estimating an initial guess of the correct root. This GEO example has one positive root. However, it was stated in [2] as follows: "Descartes' Rule of Signs indicates three positive real roots for Gauss; two with the LEO ranges and one GEO range. However, the Gauss geometric method deduced a range near GEO, and therefore the correct range is that of the GEO." This was a misleading, incorrect statement because this example has only one positive root, as shown in this paper.

Plots of $f(x)$ in blue and $g(x) = x^8 + ax^6$ in red for Example 8 (Saturn) with $\psi = 85$ deg are provided in Figure 6. It is interesting to note that this example has three non-spurious roots. As can

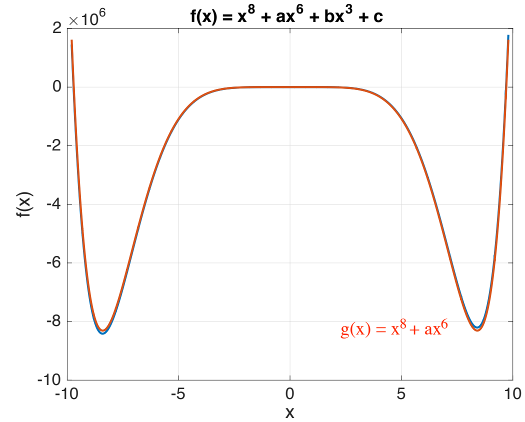

Figure 6. Example 8 Heliocentric Saturn with Gauss coefficients [2].

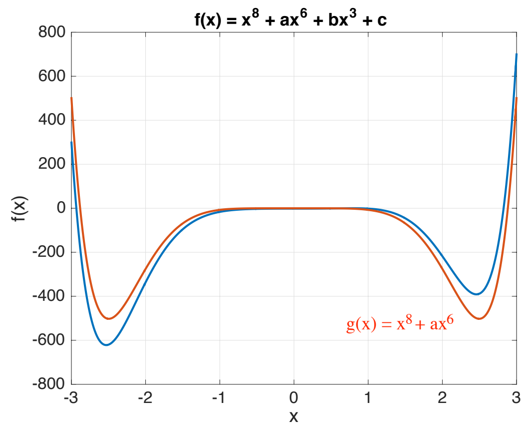

Figure 7. Example 10 Heliocentric Ceres with Gauss coefficients [2].

be noticed in this figure, $g(x) = x^8 + ax^6$ is an excellent approximation of $f(x)$, which can be used for quickly estimating an initial guess of the correct root and also for selecting the correct root from the three non-spurious roots that have been already obtained.

Plots of $f(x)$ in blue and $g(x) = x^8 + ax^6$ in red for Example 10 (Ceres) with Gauss coefficients and with $\psi = 165$ deg are provided in Figure 7. Similar to Example 8 (Saturn), this example has also three positive roots; however, two of the three positive roots are the spurious roots. Although the plot of $g(x) = x^8 + ax^6$ is not nearly identical to the $f(x)$ plot, $g(x)$ can be used for quickly estimating an initial guess of the correct root.

Figure 8 shows plots of $f(x)$ and $g(x) = x^8 + ax^6$ for Example 10 (Ceres) with Laplace coefficients. Due to numerical error, we have a trivial spurious root of $x = 1.059854840360126$ instead of $x = 1$. However, this example has one non-spurious root.

Consider another example with three positive roots [7] described by

$$\begin{aligned} x^2 &= y^2 + 1.1493y + 0.9734 \\ y &= \tilde{A} + \frac{\tilde{B}}{x^3} = 2.703 - \frac{2.596}{x^3} \end{aligned}$$

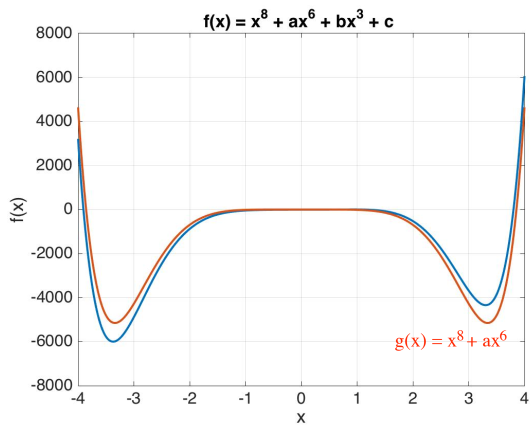

Figure 8. Example 10 Heliocentric Ceres with Laplace coefficients [2].

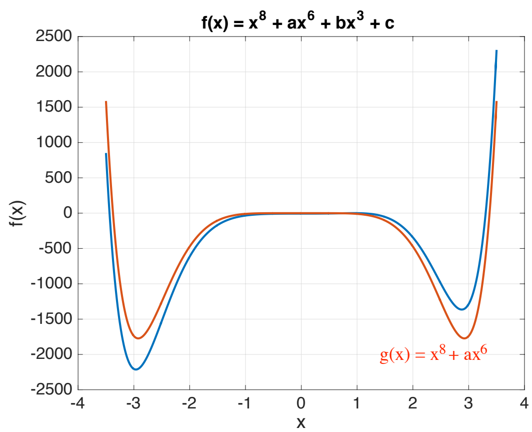

Figure 9. Danby's Example with Laplace coefficients [7].

It can be easily verified that $x = \sqrt{0.9734} = 0.98661$ is a trivial spurious solution ($y = 0$) and that $C = 0.57465$, $\psi = 125.62$ deg, and $\cos\psi = -0.5824$. Its 8th order polynomial equation in x becomes

$$f(x) = x^8 - 11.38616x^6 + 17.01671x^3 - 6.73854 = 0$$

Plots of $f(x)$ and $g(x) = x^8 + ax^6$ of this example with three positive roots are provided in Figure 9. Although the $g(x)$ plot is not nearly identical to the $f(x)$ plot, $g(x)$ can be used for quickly estimating an initial guess of the correct root ($x = 3.3045, y = 2.631, \phi = 14.4$ deg). This example has one non-spurious solution.

For the examples with three positive roots examined herein, the correct root can be easily determined by considering the root of $g(x) = x^8 + ax^6 = x^6(x^2 + a)$. The largest positive root of each example happened to be the correct root, and an example having the smallest or the intermediate positive root as the correct root still needs to found.

CONCLUSIONS

A classical yet still mystifying problem, which is concerned with multiple solutions of the angles-only initial orbit determination (IOD) polynomial equations of Laplace and Gauss, has been reexamined in this paper. A very simple method of determining the correct root from two or three non-spurious roots has been presented. The simplicity of the proposed method, utilizing an approximate polynomial equation of the form: $g(x) = x^8 + ax^6 = 0$ but without requiring any *a priori* knowledge and/or additional observations of the object, has been demonstrated. An approximate polynomial equation, either $g(x) = x^8 + c = 0$ or $g(x) = x^8 + ax^6 = x^6(x^2 + a) = 0$, can also be used for quickly estimating an initial guess of the correct root. For the examples with three positive roots examined in this paper, the largest positive root of each example happened to be the correct root, and thus an example having the smallest or the intermediate positive root as the correct root still needs to be found and further studied.

ACKNOWLEDGEMENT

The authors would like to express special thanks to Dr. Gim Der who has provided them with invaluable advices and numerous references on the multiple-root problem of the angles-only initial orbit determination polynomial equations of Laplace and Gauss.

APPENDIX

The Laplace Method [11, 13]

For the angles-only IOD problem illustrated in Figures 1 and 2, we have $\mathbf{r} = \rho\mathbf{L} + \mathbf{R}$ and $\ddot{\mathbf{r}} = -\mu\mathbf{r}/r^3$ resulting in

$$\ddot{\mathbf{r}} = 2\dot{\rho}\dot{\mathbf{L}} + \ddot{\rho}\mathbf{L} + \rho\ddot{\mathbf{L}} + \ddot{\mathbf{R}} = -\mu\frac{\mathbf{r}}{r^3} \tag{66}$$

Equation (66) can be rearranged as

$$\mathbf{L}\ddot{\rho} + 2\dot{\mathbf{L}}\dot{\rho} + \left(\ddot{\mathbf{L}} + \frac{\mu}{r^3}\mathbf{L}\right)\rho = -\left(\ddot{\mathbf{R}} + \frac{\mu}{r^3}\mathbf{R}\right) \tag{67}$$

or

$$\begin{bmatrix} \mathbf{L} & 2\dot{\mathbf{L}} & \ddot{\mathbf{L}} + (\mu/r^3)\mathbf{L} \end{bmatrix} \begin{bmatrix} \ddot{\rho} \\ \dot{\rho} \\ \rho \end{bmatrix} = -\begin{bmatrix} \ddot{\mathbf{R}} + (\mu/r^3)\mathbf{R} \end{bmatrix} \tag{68}$$

Using Lagrange's interpolation formula, we can obtain general analytical expressions of $\mathbf{L}$, $\dot{\mathbf{L}}$, and $\ddot{\mathbf{L}}$, as follows:

$$\mathbf{L}(t) \approx \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}\mathbf{L}_1 + \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}\mathbf{L}_2 + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}\mathbf{L}_3 \tag{69}$$

$$\dot{\mathbf{L}}(t) \approx \frac{2t-t_2-t_3}{(t_1-t_2)(t_1-t_3)}\mathbf{L}_1 + \frac{2t-t_1-t_3}{(t_2-t_1)(t_2-t_3)}\mathbf{L}_2 + \frac{2t-t_1-t_2}{(t_3-t_1)(t_3-t_2)}\mathbf{L}_3 \tag{70}$$

$$\ddot{\mathbf{L}}(t) \approx \frac{2}{(t_1-t_2)(t_1-t_3)}\mathbf{L}_1 + \frac{2}{(t_2-t_1)(t_2-t_3)}\mathbf{L}_2 + \frac{2}{(t_3-t_1)(t_3-t_2)}\mathbf{L}_3 \tag{71}$$

where $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ are the known line-of-sight vectors to the object at the three observation times.

For heliocentric or space-based geocentric observations, we have $\ddot{\mathbf{R}} = -\mu\mathbf{R}/R^3$. For ground-based geocentric observations, we may use $\ddot{\mathbf{R}} = \boldsymbol{\omega}_\oplus \times (\boldsymbol{\omega}_\oplus \times \mathbf{R})$ or Lagrange's interpolation formula to estimate $\ddot{\mathbf{R}}$, as follows:

$$\mathbf{R}(t) \approx \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}\mathbf{R}_1 + \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}\mathbf{R}_2 + \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}\mathbf{R}_3 \tag{72}$$

$$\dot{\mathbf{R}}(t) \approx \frac{2t-t_2-t_3}{(t_1-t_2)(t_1-t_3)}\mathbf{R}_1 + \frac{2t-t_1-t_3}{(t_2-t_1)(t_2-t_3)}\mathbf{R}_2 + \frac{2t-t_1-t_2}{(t_3-t_1)(t_3-t_2)}\mathbf{R}_3 \tag{73}$$

$$\ddot{\mathbf{R}}(t) \approx \frac{2}{(t_1-t_2)(t_1-t_3)}\mathbf{R}_1 + \frac{2}{(t_2-t_1)(t_2-t_3)}\mathbf{R}_2 + \frac{2}{(t_3-t_1)(t_3-t_2)}\mathbf{R}_3 \tag{74}$$

By applying Cramer's rule to Eq. (68), as discussed in [11,13], we obtain the dynamical equation of the Laplace method as

$$\rho = A + \frac{B}{r^3} \tag{75}$$

where

$$\begin{aligned}
A &= -\frac{2D_1}{D} \quad \text{if } D \neq 0 \\
B &= -\frac{2\mu D_2}{D} \quad \text{if } D \neq 0 \\
D &= \det\left[\begin{array}{ccc} \mathbf{L} & 2\dot{\mathbf{L}} & \ddot{\mathbf{L}} + (\mu/r^3)\mathbf{L} \end{array}\right] = 2\det\left[\begin{array}{ccc} \mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{L}} \end{array}\right] \\
D_1 &= \det\left[\begin{array}{ccc} \mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{R}} \end{array}\right] \\
D_2 &= \det\left[\begin{array}{ccc} \mathbf{L} & \dot{\mathbf{L}} & \mathbf{R} \end{array}\right]
\end{aligned}$$

For ground-based geocentric observations, we have $D_1 = 0$ and $A = 0$ if we can ignore the effect of Earth's rotation. For heliocentric or space-based geocentric observations, we have

$$D_1 = \det\left[\begin{array}{ccc} \mathbf{L} & \dot{\mathbf{L}} & \ddot{\mathbf{R}} \end{array}\right] = \det\left[\begin{array}{ccc} \mathbf{L} & \dot{\mathbf{L}} & -(\mu/R^3)\mathbf{R} \end{array}\right] = -(\mu/R^3)D_2 \tag{76}$$

and Eq. (75) can be written as

$$\rho = \frac{2\mu D_2}{D}\left(\frac{1}{R^3} - \frac{1}{r^3}\right) = \frac{2\mu D_2}{DR^3}\left(1 - \frac{R^3}{r^3}\right) \tag{77}$$

which has the form of Eq. (12) or Eq. (22), first suggested by Lagrange in 1778 [4].

Combining Eq. (75) with the Sun-Earth-Object triangle equation of the form

$$r^2 = \rho^2 + 2C\rho + R^2 \tag{78}$$

where $C = \mathbf{L}_2 \cdot \mathbf{R}_2$ at t_2, we obtain an 8th order polynomial equation of Laplace as

$$f(r) = r^8 - (A^2 + 2AC + R^2)r^6 - 2B(A+C)r^3 - B^2 = 0 \tag{79}$$

where

$$A = \frac{2\mu D_2}{DR^3}; \qquad B = -\frac{2\mu D_2}{D} \tag{80}$$

for heliocentric or space-based geocentric observations. *Note that $r = R$ is a trivial spurious solution of the Laplace method for heliocentric or space-based geocentric observations.*

Once r and ρ are determined at $t = t_2$, $\dot{\rho}$ can be determined as [11, 13]

$$\dot{\rho} = -\frac{D_3}{D} - \frac{\mu D_4}{D}\frac{1}{r^3} \quad \text{if } \; D \neq 0 \tag{81}$$

where

$$D_3 = \det\begin{bmatrix} \mathbf{L} & \ddot{\mathbf{R}} & \ddot{\mathbf{L}} \end{bmatrix} = \det\begin{bmatrix} \mathbf{L} & -(\mu/R^3)\mathbf{R} & \ddot{\mathbf{L}} \end{bmatrix} = -(\mu/R^3)D_4 \tag{82}$$

$$D_4 = \det\begin{bmatrix} \mathbf{L} & \mathbf{R} & \ddot{\mathbf{L}} \end{bmatrix} \tag{83}$$

The velocity vector $\mathbf{v}$ at $t = t_2$ can then be determined as

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\rho}\mathbf{L} + \rho\dot{\mathbf{L}} + \dot{\mathbf{R}} \tag{84}$$

where $\dot{\mathbf{R}}$ at $t = t_2$ can be estimated from Eq. (73).

The Gauss Method [13, 15]

Due to the fact that three heliocentric position vectors lie in the same orbital plane, we have

$$c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = 0 \tag{85}$$

where the coefficient c_2 can be assumed as $c_2 = -1$. Consequently, we have

$$\mathbf{r}_1 \times \mathbf{r}_3 c_1 = \mathbf{r}_2 \times \mathbf{r}_3 \tag{86}$$

$$\mathbf{r}_1 \times \mathbf{r}_3 c_3 = \mathbf{r}_1 \times \mathbf{r}_2 \tag{87}$$

The position vector $\mathbf{r}_1$ and $\mathbf{r}_3$ are expressed as

$$\mathbf{r}_1 = f_1\mathbf{r}_2 + g_1\mathbf{v}_2 \tag{88}$$

$$\mathbf{r}_3 = f_3\mathbf{r}_2 + g_3\mathbf{v}_2 \tag{89}$$

where f_i and g_i are the Lagrange coefficients approximated as

$$f_1 \approx 1 - \frac{\tau_1^2}{2}\frac{\mu}{r_2^3}; \quad g_1 \approx \tau_1 - \frac{\tau_1^3}{6}\frac{\mu}{r_2^3} \tag{90}$$

$$f_3 \approx 1 - \frac{\tau_3^2}{2}\frac{\mu}{r_2^3}; \quad g_3 \approx \tau_3 - \frac{\tau_3^3}{6}\frac{\mu}{r_2^3} \tag{91}$$

where $\tau_1 = t_1 - t_2$ and $\tau_3 = t_3 - t_2$. We then obtain

$$(f_1\mathbf{r}_2 + g_1\mathbf{v}_2) \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2)c_1 = \mathbf{r}_2 \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2) \tag{92}$$

$$(f_1\mathbf{r}_2 + g_1\mathbf{v}_2) \times (f_3\mathbf{r}_2 + g_3\mathbf{v}_2)c_3 = (f_1\mathbf{r}_2 + g_1\mathbf{v}_2) \times \mathbf{r}_2 \tag{93}$$

which result in

$$c_1 = \frac{g_3}{f_1 g_3 - f_3 g_1} \approx \frac{\tau_3}{\tau_3 - \tau_1} + \frac{\tau_3((\tau_3 - \tau_1)^2 - \tau_3^2)}{6(\tau_3 - \tau_1)}\frac{\mu}{r_2^3} = a_1 + b_1\frac{\mu}{r_2^3} \tag{94}$$

$$c_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1} \approx \frac{-\tau_1}{\tau_3 - \tau_1} - \frac{\tau_1((\tau_3 - \tau_1)^2 - \tau_1^2)}{6(\tau_3 - \tau_1)}\frac{\mu}{r_2^3} = a_3 + b_3\frac{\mu}{r_2^3} \tag{95}$$

Substituting $\mathbf{r}_i = \rho_i \mathbf{L}_i + \mathbf{R}_i$ into Eq. (85), we obtain

$$\begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 \end{bmatrix} \begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \end{bmatrix} \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix} \tag{96}$$

or

$$\begin{bmatrix} c_1\rho_1 \\ c_2\rho_2 \\ c_3\rho_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \end{bmatrix} \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix} \tag{97}$$

From this equation, we obtain the dynamical equation of the Gauss method for ρ_2 as

$$\rho_2 = A + \frac{B}{r_2^3} \tag{98}$$

where

$$\begin{aligned} A &= M_{21}a_1 - M_{22} + M_{23}a_3 \\ B &= \mu(M_{21}b_1 + M_{23}b_3) \end{aligned}$$

and M_{ij} is the ij-th element of the 3×3 matrix: $\begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \mathbf{L}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \mathbf{R}_3 \end{bmatrix}$.

Note that $\ddot{\mathbf{R}}$ was never used in obtaining Eq. (98) of the Gauss method and that the coefficient A doesn't explicitly contain the gravitational parameter μ, contrary to the Laplace method. Both A and B coefficients of the Laplace method explicitly contain μ for heliocentric or space-based geocentric observations, as can be noticed in Eq. (80).

Finally, an 8th order polynomial equation of Gauss can be obtained as

$$f(r) = r^8 - (A^2 + 2AC + R^2)r^6 - 2B(A + C)r^3 - B^2 = 0 \tag{99}$$

where $r = r_2$, $R = R_2$, and $C = \mathbf{L}_2 \cdot \mathbf{R}_2$ at $t = t_2$.

In terms of $x = r/R$ and $y = \rho/R$, the 8th order dimensionless polynomial equation can be expressed as

$$f(x) = x^8 + ax^6 + bx^3 + c = 0 \tag{100}$$

where

$$\begin{aligned} a &= -(A^2 + 2AC + R^2)/R^2 \\ b &= -2B(A + C)/R^5 \\ c &= -B^2/R^8 \end{aligned}$$

and the dimensionless dynamical equation is expressed as

$$y = \tilde{A} + \frac{\tilde{B}}{x^3} \tag{101}$$

where $\tilde{A} = A/R$ and $\tilde{B} = B/R^4$.

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